# Duality and equilibrium

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# Duality and Equilibrium



T equilibrium theory
V duality

Research memorandum



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Duality and equilibrium

by

H.N. Weddepohl.

Part 1

## Introduction *)

The concept of a dual utility function was introduced by Roy [ 6] . The theory of duality was analysed with respect to preferences by Milleron [ 5] and Weddepohl [ 10] . Duality in the theory of production functions was extensively studied by Shephard [ 8] .
The mathematical concept of a dual set can be found in Eggleston [ 4] and Valentine [ 9] .

In the present paper dual sets are introduced both for preferences and for production sets. It is shown that with a preference structure in commodity space is associated a preference structure in price space and with production sets in commodity space correspond production sets in price space. Therefore also equilibria can be defined in dual space. Ruys [ 7] applied this idea to an economy with collective goods only.

In part 2 of this paper we define some duality concepts and their properties. In part 3 the theory of part 2 is applied to a direct market and the properties of a Pareto optimum, the core and of a competitive equilibrium in price space are given. In part 4 an economy with production is introduced by a set of assumptions, similar to the ones given by Debreu [ 3] . A competitive equilibrium is defined both in price space and in commodity space.
A proof for the existence of an equilibrium is given by showing its existence in price space. The advantage of this method is that a bounded set containing the solution is easily found. We believe that this method of proof could be applied to a wide range of problems in equilibrium analysis.

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*) I thank Mr.P.Ruys for his comments and his helpful discussion.

## Part 2

### 2.1 Cones and related sets

In this paper we only use cones, aureoled sets and star shaped sets with respect to the origin. They can however be defined with respect to any point. (See Berge p. 15)

### 2.1.1 Def.

1) A set $K \subset R^n$ is called a cone (with respect to the origin), if

$$x \in K \text{ and } \lambda \geq 0 \Rightarrow \lambda x \in K$$

2) A set $K \subset R^n$ is called star-shaped (with respect to 0), if

$$x \in K \text{ and } 1 \geq \lambda \geq 0 \Rightarrow \lambda x \in K$$

3) A set $K \subset R^n$ is called aureoled (with respect to 0), if

$$x \in K \text{ and } \lambda \geq 1 \Rightarrow \lambda x \in K$$

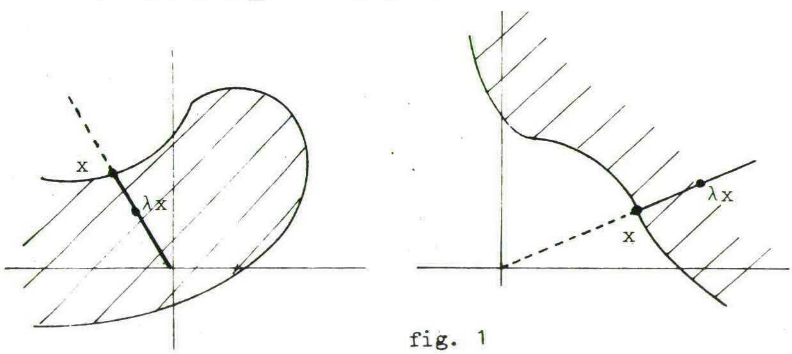



fig. 1

Obviously a cone is star-shaped as well as aureoled.

### 2.2 Closures

With any set in $R^n$ can be associated a set of a certain type, which is the smallest set of that type that contains the original set. Such an operation is called a closure. A closure can also be considered as a mapping $f : \mathfrak{X} \to \mathfrak{X}$, if $\mathfrak{X}$ is the set of all subsets of $R^n$. Now we have a closure if (see Berge, p. 13), for $K \subset R^n$

$$K \subset f(K)$$

$$C \subset K \Rightarrow f(C) \subset f(K)$$

$$f(f(K)) = f(K)$$

$$f(\emptyset) = \emptyset$$

We shall use five closures: the ordinary (topological) closure, the convex hull, the star-closure, the cone of a set and the aureoled closure. It is obvious that these are closured in the above sense

2.2.1 Def.

Given a set $K \subset R^n$, we have the following closures

a. (topological)

$$\mathrm{Cl}\ K = \{x \in R^n \mid \forall\ \varepsilon > 0 : B_\varepsilon(x) \cap K \neq \emptyset\}$$

b. (convex hull)

$$\mathrm{Conv}\ K = \{x \in R^n \mid \exists\ x_i \in K; \exists\ \alpha_i \geq 0: x = \Sigma\ \alpha_i x_i \text{ and } \Sigma\ \alpha_i = 1\}$$

c. (star-closure)

$$\mathrm{St}\ K = \{x \in R^n \mid \exists\ y \in K; \exists\ \lambda: < 0 \leq \lambda \leq 1 \text{ and } x = \lambda\ y\}$$

d. (aureoled closure)

$$\mathrm{Au}\ K = \{x \in R^n \mid \exists\ y \in K; \exists\ \lambda \geq 1 : x = \lambda\ y\}$$

e. (cone)

$$\mathrm{Cone}\ K = \{x \in R^n \mid \exists\ y \in K, \exists\ \lambda \geq 0 : x = \lambda\ y\}$$

2.2.2 Property

a) If K is convex, all closures of definition are convex

b) If K is compact, all closures are closed and St K is compact

c) If K is convex we have

$$K = \mathrm{Au}\ K \cap \mathrm{St}\ K$$

d) For every K

$$\text{Cone } K = \text{Au }(\text{St } K) = \text{St }(\text{Au})K = \text{Au } K \cup \text{St } K.$$

Proof of c

Since $K \subset \text{Au } K$ and $K \subset \text{St } K$, we have $K \subset \text{Au } K \cap \text{St } K$.
Now let $x \in \text{Au } K \cap \text{St } K$. Since $x \in \text{Au } K$, there exists $\lambda \geq 1$ and $y \in K$, such that $x = \lambda\, y$ and since $x \in \text{St } K$, there exists $0 \leq \mu \leq 1$ and $z \in K$, such that $x = \mu\, z$ and

$$x = \frac{\lambda\,(1-\mu)}{\lambda-\mu}\, y + \frac{\mu(\lambda-1)}{\lambda-\mu}\, z$$

where $\lambda(1-\mu) + \mu(\lambda-1) = \lambda-\mu$, hence $x \in C$.

## 2.3 Openings

In a way similar to closures, "openings" can be defined. With any set is associated a set of a certain type, which is the largest set of this type, contained in the original set. If f represents an opening, it must hold for $K \subset R^n$

$$f(K) \subset K$$

$$C \subset K \;\Rightarrow\; f(C) \subset f(K)$$

$$f(f(K)) = f(K)$$

$$f(R^n) = R^n$$

Apart from the interior of a set, we define the interior cone, the interior star and the interior aureole. It is obvious that these are openings; (the opening of a convex set does not exist)

### 2.3.1 Def.

We have the following openings, for $K \subset R^n$

a) $\text{Int } K = \{x \mid \forall\; \varepsilon : B_\varepsilon(x) \subset K\}$

b) the interior star

$\text{Stint } K = \{x \mid 0 \leq \lambda \leq 1 \;\Rightarrow\; \lambda\, x \in K\}$

c) the interior aureole

$\text{Auint } K = \{x \mid \lambda \geq 1 \Rightarrow \lambda x \in K\}$

d) the interior cone

$\text{Coneint } K = \{x \mid \lambda \geq 0 \Rightarrow \lambda x \in K\}$

2.3.2 Properties

a) $\text{Auint } C = X \setminus \text{St}(X/C)$

b) $\text{Stint } C = X \setminus \text{Au}(X \setminus C)$

c) $\text{Coneint } C = X \setminus \text{Cone } (X \setminus C)$

Proof

a) $\text{St } X \setminus C = \{x \mid \exists\, y \notin C; \exists\, \lambda \leq 1 : \lambda y = x\}$.

Let $x \in \text{Au int } C$. Suppose $x \in \text{St } X \setminus C$. Now for some $y \notin C$ and $\lambda \leq 1$, $x = \lambda y$, and we have $\frac{1}{\lambda} \geq 1$, hence $y = \frac{1}{\lambda} x \notin C$ and that is a contradiction.

Let $x \notin \text{Au int } C$. Hence for some $\lambda \geq 1$, $y = \lambda x \notin C$ hence $x = \frac{1}{\lambda} y \in \text{St}(X \setminus C)$

2.3.3. Property

$0 \notin C \Rightarrow \text{Stint } C = \emptyset$ and $\text{Coneint} = \emptyset$

$C$ is bounded $\Rightarrow \text{Auint } C = \emptyset$ and $\text{Coneint } C = \emptyset$

2.4 Duality

Let $X = R^n$ and let $P = R^n$ be "another" Euclidean space.

With any set $C \subset X$ can be associated four different sets in P.

The dual sets

2.4.1 <u>Def</u>.

$$C_+^* = \{p \in P \mid \forall\, x \in C : p\,x \geq 1\}$$

$$C_-^* = \{p \in P \mid \forall\, x \in C : p\,x \leq 1\}$$

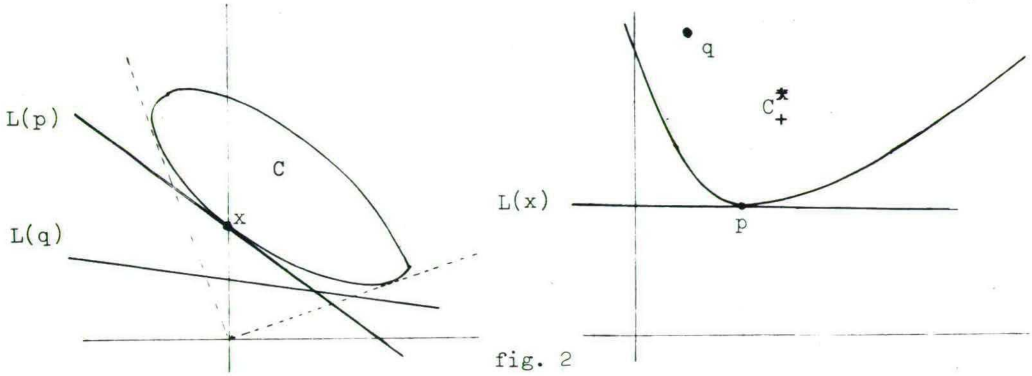



fig. 2

and the dual cones.

2.4.2 <u>Def</u>.

$$C_+^\circ = \{p \in P \mid \forall\, x \in C : p\,x \geq 0\}$$

$$C_-^\circ = \{p \in P \mid \forall\, x \in C : p\,x \leq 0\}$$

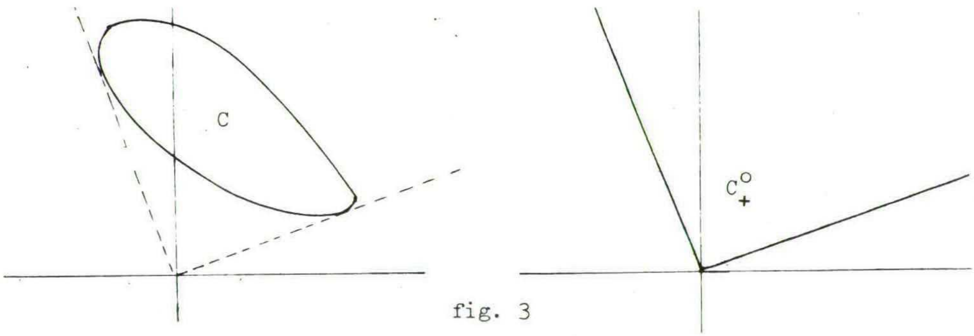



fig. 3

$C_+^*$ contains all $p \in P$, such that the hyperplanes $L(p) = \{x \mid p\,x = 1\}$ separate C and $\{0\}$. $C_-^*$ contains all $p \in P$ such that the hyperplane $L(p)$ has 0 and C on one side. $C_+^\circ$ and $C_-^\circ$ contain all p, such that the planes $\{x \mid px = 0\}$ has C on the positive and the negative side respectively.

If p is a point on the boundary of $C^*_+$, such that $\lambda p \notin C^*_+$ for $\lambda < 1$, then the hyperplane L(p) either <u>supports</u> C in some point x, or L(p) "<u>asymptotically</u>" supports C. Similarly, boundary points of $D^*_-$ either support or asymptotically support D.
(For a special type of aureoled sets duals are extensively studied in [10])

2.4.3 <u>Properties</u> of $C^*_+$

a) $C^*_+ = (\text{Int } C)^*_+ = (\text{Cl } C)^*_+ = (\text{Conv } C)^*_+ = (\text{Au } C)^*_+$

b) $C^*_+$ Is closed, convex, aureoled.

c) $B \subset C \Rightarrow B^*_+ \supset C^*_+$

d) $(B \cup C)^*_+ = B^*_+ \cap C^*_+$

e) $(B \cap C)^*_+ = \text{Conv } (B^*_+ \cup C^*_+)$

f) $0 \in \text{Cl } C \Rightarrow C^*_+ = \emptyset$

<u>Proof</u>

a) Let p be such that $\forall\, x \in C : p\, x \geq 1$. Hence also $p\, y \geq 1$ if $y \in \text{Int } C$, if $y \in \text{Cl } C$, if $y = \Sigma\, \alpha_i x_i$ for $x_i \in C$, $\alpha_i > 0$ and $\Sigma\, \alpha_i = 1$, and if $\lambda > 1$ and $y = \lambda\, x$ for $x \in C$.

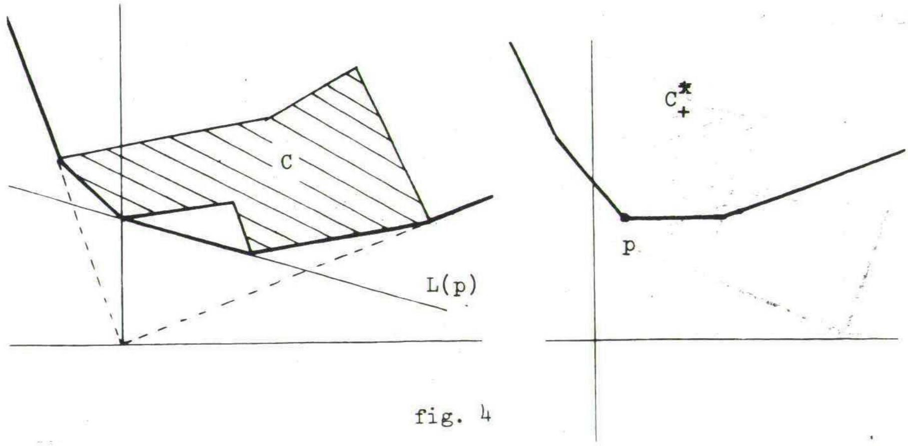



fig. 4

b) convex: $\forall\ x \in C : p\ x \geq 1$ and $q\ x \geq 1$, then $(\alpha\ p + (1-\alpha)q)\ x \geq 1$ for $\alpha \in [0,1]$; aureoled: $\forall\ x \in C : p\ x \geq 1 \Rightarrow \lambda\ p\ x \geq 1$, if $\lambda \geq 1$
closed: suppose $p \in \mathrm{Cl}\ C^{*}_{+}$ and $p \notin C^{*}_{+}$, hence there exists $x \in C$ such that $p\ x < 1$. But for $\varepsilon$ sufficiently small, also $q\ x < 1$, for $q \in B_{\varepsilon}(p)$.

c) $\forall\ x \in C : p_o\ x \leq 1 \;\Rightarrow\; \forall\ x \in B : p\ x \leq 1$

d) By c : $(B \cup C)^{*}_{+} \subset B^{*}_{+} \cap C^{*}_{+}$
Let $p \in B^{*}_{+} \cap C^{*}_{+}$, hence $\forall\ x \in B : p\ x \geq 1$ and $\forall\ x \in C : p\ x \geq 1$ and therefore also $\forall\ x \in B \cup C : p\ x \geq 1$

2.4.4 Properties of $C^{*}_{-}$

a) $C^{*}_{-} = (\mathrm{Int}\ C)^{*}_{-} = (\mathrm{Cl}\ C)^{*}_{-} = (\mathrm{Conv}\ C)^{*}_{-} = (\mathrm{St}\ C)^{*}_{-}$

b) $C^{*}_{-}$ is closed, convex, star shaped

c) $B \subset C \;\Rightarrow\; B^{*}_{-} \supset C^{*}_{-}$

d) $(B \cup C)^{*}_{-} = B^{*}_{-} \cap C^{*}_{-}$

e) $(B \cap C)^{*}_{-} = \mathrm{Conv}\ (B \cup C)^{*}_{-}$

f) $0 \in C^{*}_{-}$

g) $0 \in \mathrm{Int}\ C \;\Leftrightarrow\; C^{*}_{-}$ is bounded

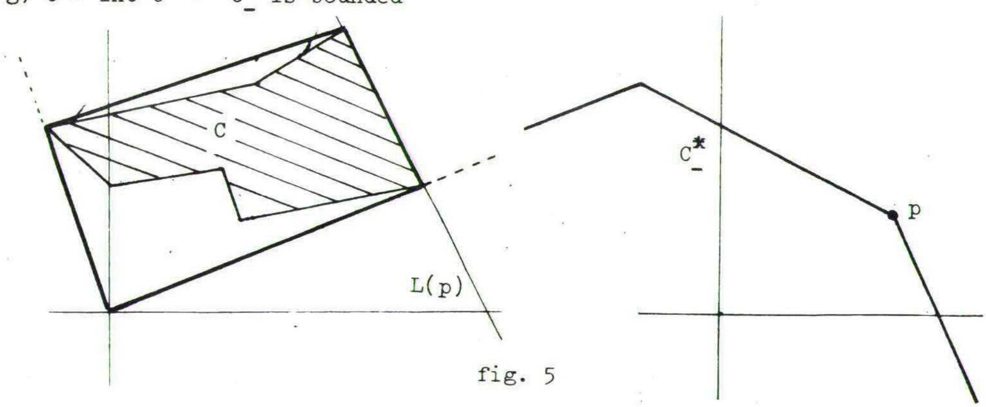



fig. 5

2.4.5 <u>Properties</u> of $C_+^{\circ}$ and $C_+^{*}$ for $C_+^{*} \neq \emptyset$

a) $C_+^{*} \subset C_+^{\circ}$

b) $C_+^{\circ} = (\text{Cone } C)_+^{\circ}$

c) $C_+^{\circ} = \text{Cl Cone } C_+^{*}$

<u>Proof</u>

a) $\forall\, x \in C : p\,x \geq 1 \;\Rightarrow\; \forall\, x \in C : p\,x \geq 0$

b) $\forall\, x \in C : p\,x \geq 1$ and $y = \lambda\, x$ for $\lambda > 0 \;\Rightarrow\; p\,y \geq 0$

c) Since $C_+^{*} \subset C_+^{\circ}$ , $\text{Cl Cone } C_+^{*} \subset C_+^{\circ}$

Suppose $p_o \in C_+^{\circ}$,

$\text{Cone } C_+^{*} = \{p \mid \exists\, \lambda > 0, \forall\, x \in C : \lambda\, p\, x \geq 1\} = \{p \mid \exists \eta, \forall\, x \in C : p\, x > \eta\}$

Now let $q \in \text{Cone } C_+^{*}$

then if $0 < \alpha \leq 1$, $\alpha\, p + (1-\alpha) p_o \in \text{Cone } C_+^{*}$, so $p_o \in \text{Cl Cone } C_+^{*}$

2.4.6 <u>Properties of</u> $C_-^{\circ}$ and $C_-^{*}$

a) $C_-^{*} \supset C_-^{\circ}$

b) $C_-^{\circ} = (\text{Cone } C)_-^{\circ}$

c) $C_-^{\circ} = (\text{Cone } C)_-^{*}$

d) $C_-^{*} \subset (\text{Coneint } C)_-^{\circ}$

<u>Proof</u>

a) $p \in C_-^{*}$, hence $\forall\, x \in C : p\, x \leq 1$

Suppose $p \notin (\text{Coneint } C)_-^{\circ}$, hence for some $x \in \text{Coneint } C$, we have

$p\, x = \alpha > 0$, hence $p(\frac{2}{\alpha}\, x) = 2$ and that is a contradiction.

2.4.7 Property

a. If C is an aureoled, closed, convex set, such that $0 \notin C$

$$(C_+^*)_+^* = C$$

e) if D is a star shaped, closed, convex set such that $0 \in C$

$$(D_-^*)_-^* = D$$

Proof (a)

1) Let $x_o \in C$

Since $p \in C_+^* \Leftrightarrow \forall x \in C : p\,x \geq 1$

we have

$$\forall p \in C^* : p\,x_o \geq 1$$

and so

$$x_o \in (C_+^*)_+^* = \{x \mid \forall p \in C^* : p\,x \geq 1\}$$

2) Let $x_o \notin C$. Let $T = \{y \mid y = \alpha\, x_o \text{ for } 0 \leq \alpha \leq 1\}$

T is compact, and since C is aureoled $T \cap C = \emptyset$. Hence there exists a hyperplane L(p) strictly separating T and C and $p \in C_+^*$.

Now $x_o \notin (C_+^*)_+^*$, since $p\,x_o < 1$

2.4.8 Theorem

Let C be a closed, convex, aureoled set, such that $0 \notin C$, and D a closed, convex star-shaped set, such that $0 \in \text{Int } D$.

a) $\exists\, x, \exists\, \lambda \neq 1 : x \in C \cap D$ and $\lambda\, x \in C \cap D \Leftrightarrow C_+^* \cap D_-^* = \emptyset$

b) $C \cap D \neq \emptyset \Rightarrow [p \in C_+^* \cap D_-^*$ and $\lambda \neq 1 \Rightarrow \lambda\, p \notin C_+^* \cap D_-^*]$

c) If Coneint $D \cap$ Cl Cone $C = \{0\}$

$C \cap D = \emptyset \Leftrightarrow \exists\; p, \exists\, \lambda \neq 1 : p \in C_+^* \cap D_-^*$ and $\lambda\, p \in C_+^* \cap D_-^*$

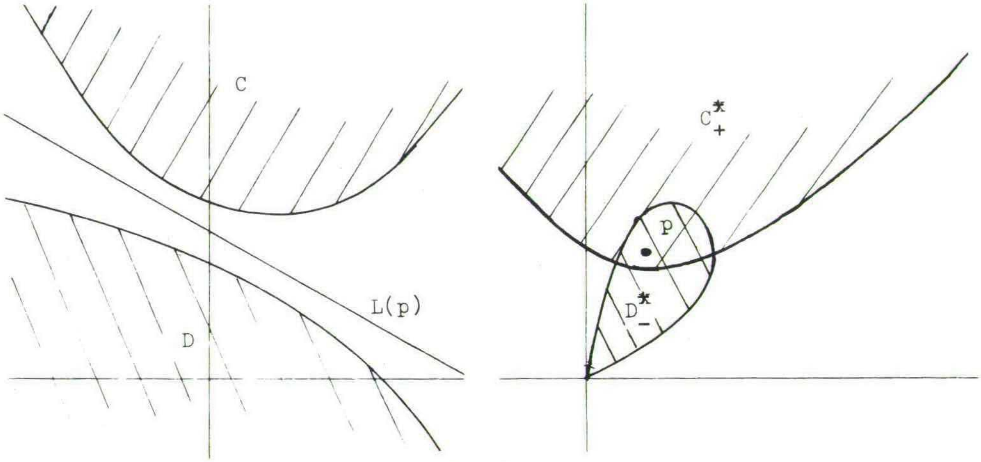



fig. 6

Proof

a) $\Rightarrow$ $p \in C^*_+ \Rightarrow p\,\frac{1+\lambda}{2}\,x > 1$, since $p\,x > 1$ and $p\,\lambda\,x > 1$; $q \in D^*_- \Rightarrow$

$q\,\frac{1+\lambda}{2}\,x < 1$, since $q\,x < 1$ and $q\,\lambda\,x < 1$.

Hence $L(\frac{1+\lambda}{2}p)$ strictly separates $C^*_+$ and $D^*_-$.

$\Leftarrow$ Let $C^*_+ \cap D^*_- = \emptyset$. Since $D^*_-$ is bounded, there exists a hyperplane $L(x)$ strictly separating both sets. Hence for som $\lambda \neq 1$ both x and $\lambda\,x$ are in $C \cap D$.

b) Let $x \in C \cap D$ and $p \in C^*_+ \cap D^*_-$, hence $px = 1$.

It is impossible that $y \in C \Rightarrow \lambda\,p\,y \geq 1$ and $y \in D \Rightarrow \lambda\,p\,y \leq 1$.

c) $\Leftarrow$ by a

$\Rightarrow$ Let $T = D \cap \text{Cl Cone } C$. T is convex compact because of the condition. T and C can be strictly separated by some plane L(p), hence $\forall\ x \in T : p\,x < 1$ and $\forall\,x \in C : p\,x > 1$. So for some $\alpha < 1$ also $\forall\,x \in T : p\,x < 1$ and $\forall\,x \in C : p\,x > \alpha$. Therefore $T \cap \frac{1}{\alpha}\,C = \emptyset$ and hence also $\frac{1}{\alpha}\,C \cap D = \emptyset$, since $\frac{1}{\alpha}\,C \subset \text{Cl Cone } C$. Now $\frac{1}{\alpha}\,C$ and D can be separated by some plane L(q) and so $L(\frac{1}{\alpha}\,q)$ and L(q) both separate C and D.

## 2.5 The dual of a sum of sets

### 2.5.1 Theorem

Let $C_i$ be a family of n closed, convex, aureoled sets such that $0 \notin C_i$ and $0 \notin \Sigma C_i$. Let $C = \Sigma C_i$.

$C^*_{i+}$ are the duals of $C_i$ and $C^*$ is the dual of $C$.

Now

$$C^*_+ = \mathrm{Cl}\ \{p \in R^n \mid \exists\ p \in C^*_i, \exists\ \alpha_i > 0 : p = \alpha_i p_i \text{ and } \Sigma\alpha_i \geq 1\}.$$

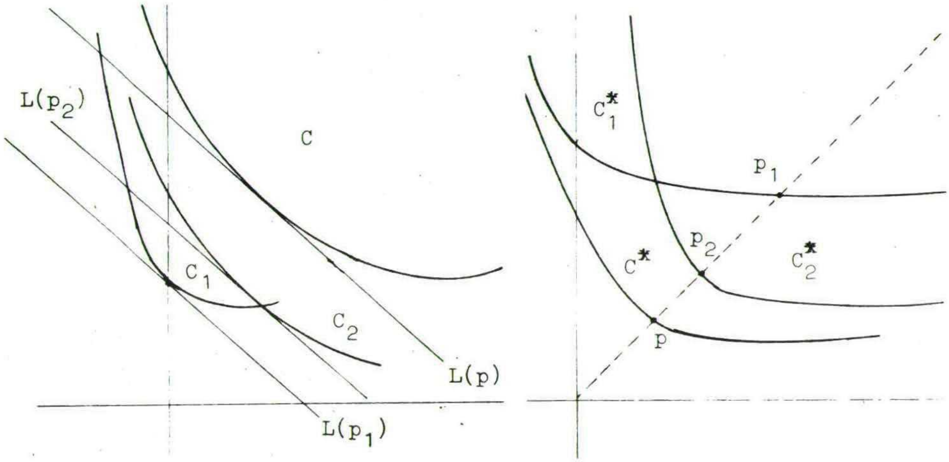



fig. 7

Proof

Let the left hand set be C'

1) $C' \subset C^*_+$.

Choose $p \in C'$ and $p = \alpha_i p_i$ for $p_i \in C^*_i$, $\Sigma\ \alpha_i \geq 1$, $\alpha_i \geq 0$.

Since $p_i \in C^*_i$, we have

$$x_i \in C_i \Rightarrow p_i x_i \geq 1$$

For all $x \in C$, there exists $x_i \in C_i$ such that $x = \Sigma\ x_i$ and now

$$p\ x = \Sigma\ p\ x_i = \Sigma\ \alpha_i\ p_i\ x_i \geq \Sigma\ \alpha_i \geq 1$$

hence $p \in C^*$

2) Int $C^* \subset C'$.

Let $p \in \text{Int } C^*$, hence $x \in C \Rightarrow p\,x > 1$

Since $p \in \text{Int } C^o_{i+}$, we have

$$\min_{x_i \in C_i} p\,x_i = p\,\bar{x}_i = \alpha_i > 0 \text{ and } \frac{1}{\alpha_i}\; p = p_i \in C_i^*$$

and now $p = \alpha_i\, p_i$, $p_i \in C_i^*$ and $\Sigma\, \alpha_i \geq 1$

for $\Sigma\, \bar{x}_i \in C$, hence $p\, \Sigma\, x_i = \Sigma\, \alpha_i \geq 1$.

A similar theorem is true for star shaped sets.
The following theorem follows directly from theorems 2.4.8 and 2.5.1.

2.5.2 <u>Theorem</u>

Let $C_i$ $(i = 1,2,\ldots,n)$ be <u>closed</u>, <u>convex</u>, <u>aureoled</u> sets and $0 \notin C_i$ and $C = \Sigma\, C_i$, such that $0 \notin \Sigma\, C_i$. D is <u>closed</u> <u>star shaped</u>, <u>convex</u> set, such that $0 \in \text{Int } D$.

a) $C \cap \text{Int } D \neq \emptyset \;\Leftrightarrow\; \forall\, p \in D^*_- : (p_i = \frac{1}{\alpha_i}\, p \in C_i^* \text{ and } \alpha_i > 0 \Rightarrow \Sigma\, \alpha_i < 1)$

b) $C \cap \text{Int } D = \emptyset$ and $C \cap D \neq \emptyset \;\Leftrightarrow$

$$\exists\, p \in D^*_- : p_i = \frac{1}{\alpha_i}\, p \in C_i^*,\ \alpha_i > 0 \text{ and } \Sigma\, \alpha_i = 1$$

$$\forall\, p \in D^*_- : (p_i = \frac{1}{\alpha_i}\, p \in C_i^*,\ \alpha_i > 0 \;\Rightarrow\; \Sigma\, \alpha_i \geq 1)$$

c) For Coneint $D \cap \text{Cl Cone } C = \{0\}$

$$C \cap D = \emptyset \Leftrightarrow \exists\, p \in D^*_- : p = \frac{1}{\alpha_i}\, p \in C_i^*,\ \alpha_i > 0 \text{ and } \Sigma\, \alpha_i > 1$$

## Part 3

### 3.1 Duality in a direct market.

Let $(I, X, w_i, \succsim_i)$ be a direct market, where $I = \{1,2,\ldots,m\}$ is the set of consumers, $X = R^{n+}$ is the commodity space, $w_i \geq 0$ are the vectors of primary resources of each individual and $\succsim_i$ is the individual's preference relation, which is assumed transitive, complete, continuous, convex and monotonous.

Let $C_i(x_i) = \{y_i \mid y_i \succsim_i x_i\}$ be a preference set of i, and this set is closed, convex and aureoled.

Let $\mathcal{A}$ be the set of allocations, where $x = (x_i)$

$$\mathcal{A} = \{x \mid \Sigma\, x_i = \Sigma\, w_i\}$$

and we have

$$x \in \mathcal{A} \Rightarrow \Sigma\, w_i = \Sigma\, x_i \in \Sigma\, C_i(x_i)$$

$\mathcal{P}$ is the set of Pareto optima

$$\mathcal{P} = \{x \in \mathcal{A} \mid y \in \mathcal{A} \Rightarrow \exists i: x_i \succ_i y_i \text{ or } \forall i: x_i \sim y_i\}.$$

Now it can be shown that

$$x \in \mathcal{P} \Rightarrow \Sigma\, x_i \in \text{Bnd}\, \Sigma\, C_i\, (x_i)$$

and

$$x \in \mathcal{P} \Leftrightarrow \exists\, p \in R^{n+} : y_i \in C_i(x_i) \Rightarrow p\, y_i \geq p\, x_i$$
$$\text{and } y_i \in \text{Int}\, C_i\, (x_i) \Rightarrow p\, y_i > p\, x_i$$

or equivalently.

$L(p)$ supports $\Sigma\, C_i\, (x_i)$ in $\Sigma\, x_i$ and $L\,(\frac{1}{\Sigma\, p\, x_i}\, p)$ supports $C_i(x_i)$ in $x_i$.

For any $S \subset I$, the <u>core</u> $\mathcal{C}$ is the set

$$\mathcal{C} = \{x \in \mathcal{A} \mid (y \in \mathcal{A} \text{ and } \sum_s y_i = \sum_s w_i) \Rightarrow$$

$$(\exists_i \in S : x_i \succ_i y_i \text{ or } {}_i \in S : x_i \sim y_i)\}$$

It can be shown that

$$x \in \mathcal{C} \Leftrightarrow \forall\, S \subset I : \sum_s w_i \notin \text{Int} \sum_s C_i(x_i)$$

$$\mathcal{C} \subset \mathcal{P}$$

$$x \in \mathcal{C} \Leftrightarrow 1^\circ\ \exists\, p : y_i \in C_i(x_i) \Rightarrow p\, y_i \geq p\, x_i$$

$$\text{and } y_i \in \text{Int } C_i(x_i) \Rightarrow p\, y_i > p\, x_i$$

$$2^\circ\ \forall\, S,\ \exists\, q : y \in \Sigma\, C_i{}^{(x_i)} \Rightarrow q\, y \geq q \sum_s w_i$$

A <u>competitive equilibrium</u> is an allocation of the set

$$\mathcal{E} = \{x \in \mathcal{A} \mid \exists\, p,\ \forall_i : p\, x_i = p\, w_i \text{ and}$$

$$y \in C_i(x_i) \Rightarrow p\, y_i \geq p\, w_i\}.$$

Now all these equilibrium concepts have interesting properties in <u>dual space</u>, by application of theorems 2.5.1 and 2.5.2

Let $C_i^*\, (p_i) = C_i^*\, (x_i)$ if $p_i \in \text{Bnd } C_i^*\, (x_i)$. We drop + : $C_i^*(x_i) = C_{i+}^*(x_i)$

<u>Allocation</u>

If $x \in \mathcal{A}$, is dual space

$$L(x_i) \text{ supports } C^*(x_i) \text{ in } p_i$$

$$L(\Sigma\, x_i) \cap \text{Int } (\Sigma\, C_i(x_i))^* = \emptyset.$$

Pareto optimum

If and only if $x \in \mathcal{P}$, there exist $p_i$, such that

$L(x_i)$ supports $C_i^*(x_i)$ in $p_i$

$L(\Sigma\, x_i)$ supports $(\Sigma\, C_i(x_i))^*$ in p

where $p = \alpha_i\, p_i$ and $\Sigma\, \alpha_i = 1$, hence $p\, x_i = \alpha_i\, p_i\, x_i = \alpha_i$

So in price space a Pareto optimum is characterized by a price p and an income distribution $\alpha_i$, for $\Sigma\, \alpha_i = 1$ and $p_i = \frac{1}{\alpha_i}\, p$.

Core

If and only if $x \in \mathcal{C}$, there exist p and $\alpha_i$, such that

$L(x_i)$ supports $C_i^*(x_i)$ in $p_i$

$L(\Sigma\, x_i)$ support $(\Sigma\, C_i(x_i))^*$in p

for $p = \alpha_i\, p_i$ and $\Sigma\, \alpha_i = 1$

and for every $S \subset I$

$L(\Sigma_S\, w_i) \cap (\Sigma_S\, C_i(x_i))^* \neq \emptyset$.

The last intersection contains q such that

$L(q_i)$ separates $\Sigma_S\, C_i(x_i)$ and $\Sigma_S\, w_i$.

Competitive equilibrium (fig 8, see also fig. 7)

If and only if $x \in \mathcal{E}$, there exist p and $\alpha_i$, such that

$L(x_i)$ supports $C_i^*\,(x_i)$ in $p_i$

$L(\Sigma\, x_i)$ supports $C_i(\Sigma\, C_i(x_i))^*$ in p

where $p = \alpha_i\, p_i$ and $\Sigma\, \alpha_i = 1$

and $\forall i : p_i \in L(w_i)$, (hence $p_i\, w_i = 1$ or $p\, w_i = \alpha_i$)

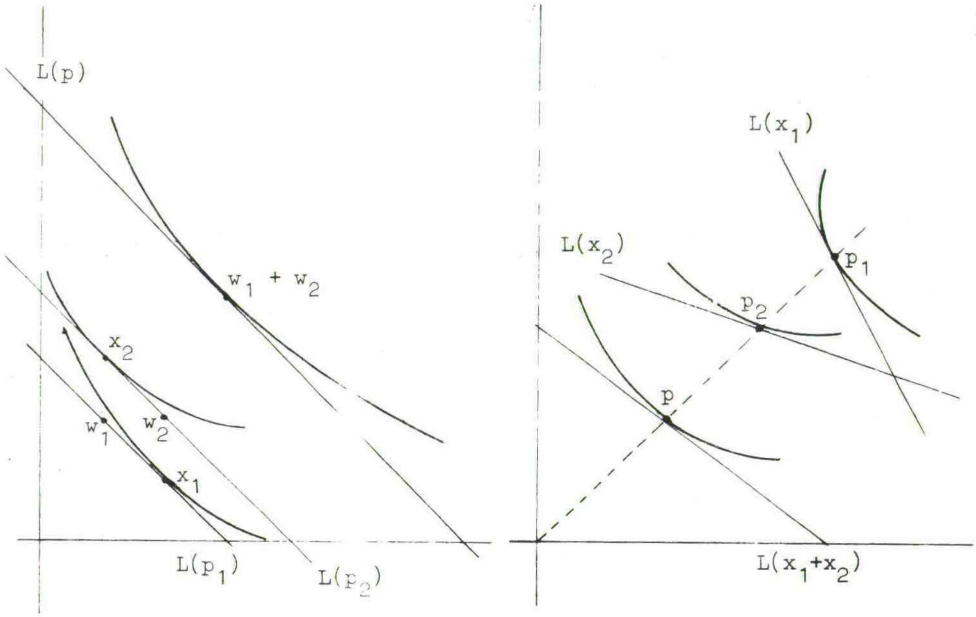



fig. 8

Part 4

4.1 Price equilibrium in price space

An economy is defined by the following concepts.

1) A set $I = \{1,2,\dots,m\}$ of consumers

2) A consumption set $X_i \subset R^n$ for each i

3) A vector of primary resources $w_i \in R^n$ for each i

4) A preference relation $\succsim_i$ for each i

5) A production set $Z \subset R^n$ (this set may be a sum of production set $Z_j$ of individual producers: $Z = \Sigma Z_j$)

6) A total production set $Y = Z + \{\Sigma w_i\}$

In the next section we shall introduce a set of assumptions, in the rest of this section however, we shall be vague about assumptions and assume implicitely that the concepts to be introduced are meaningful. Let $C_i(x_i) = \{y_i \mid y_i \succsim_i x_i\}$. Now a price equilibrium is an allocation $\overline{x}_i \in X_i$, a production $\overline{y} \in Y$ and a price vector $\overline{p} \in R^{n+}$, such that

$$\overline{p}\,\overline{y} \geq \overline{p}\,z \text{ for any } z \in Y$$

$$\overline{p}\,\overline{x}_i \leq \overline{p}\,v_i \text{ for any } v_i \in C_i(\overline{x}_i)$$

$$\overline{y} = \Sigma\,\overline{x}_i$$

If we choose $\overline{p}$ such that $\overline{p}\,\overline{y} = 1$, and if we assume that for each i $\overline{p}\,\overline{x}_i > 0$, we have

$$L(\overline{p}) \text{ supports } Y \text{ in } \overline{y}$$

$$L(\overline{p}) \text{ supports } \Sigma\, C_i(\overline{x}_i) \text{ in } \Sigma\, x_i$$

$$L(\overline{p}_i) \text{ supports } C_i(\overline{x}_i) \text{ in } \overline{x}_i, \text{ for } \overline{p}_i = \frac{1}{\overline{p}\,\overline{x}_i}\,\overline{p}\,.$$

Since $\overline{p}\ \overline{x}_i > 0$, $L(\overline{p}_i)$ also supports Au $C_i(\overline{x}_i)$ in $x_i$ en hence $L(\overline{p})$ supports $\Sigma$ Au $C_i(\overline{x}_i)$ in $\Sigma\ \overline{x}_i$.

To simplify the notation we shall denote

$$Y^* = Y_-^* \text{ and } C^*\ (x_i) = C(x_i)_+^*$$

Now $Y^*$ contains all prices at which any feasible production costs <u>at most</u> 1, whereas $C_i^*(x_i)$ contains all prices such that a consumption, which is preferred or indifferent to $x_i$, costs <u>at least</u> 1.

For the equilibrium $(\overline{x}_i, \overline{y}, \overline{p})$ we can now state in price space:

$L(\overline{y})$ supports both $Y^*$ and $(\Sigma\ C_i(\overline{x}_i))^*$ in $\overline{p}$ hence $L(\overline{y})$ separates both sets, whereas

$L(\overline{x}_i)$ supports $C_i^*(\overline{x}_i)$ in $\overline{p}_i$

We can state equivalently:

$$\overline{p}_i \in C_i^*\ (\overline{p}_i) = C_i^*(\overline{x}_i)$$

$$\overline{p} \in (\Sigma\ C_i\ (\overline{p}_i))^*$$

and

$$(\Sigma\ C_i\ (\overline{p}_i))^* \cap \text{Int } Y^* = \emptyset.$$

Now $\overline{p}\ \overline{x}_i = \lambda_i$ is the income of the i'th individual, where $\Sigma\ \lambda_i = 1$ and $\overline{p} = \lambda_i\ p_i$.

If a price equilibrium, besides the condition above also verifies that $\overline{p}_i\ \overline{x}_i$ is equal to a predetermined income, it is a <u>competitive</u> equilibrium.

## 4.2 <u>Assumptions</u>

We introduce a set of assumptions on the economy of section 4.1. These assumptions are inspired on the ones of Debreu in [ 3 ], they are however stronger, but weaker than those in [ 2 ].

Consumers

A.1 $X_i$ is closed and convex and $0 \notin X_i$

A.2 $\succsim_i$ is transitive, complete,

A.3 $C_i(x_i) = \{y_i \mid y_i \succsim_i x_i\}$ is convex

A.4 $C_i(x_i)$ and $\{y_i \mid x_i \succsim_i y_i\}$ are closed.

A.5 For each i, and for $x_i \in X_i$ and $t \in R^n$

$$[\ \forall \varepsilon,\ \exists \lambda : B_\varepsilon\ (x+\lambda t) \cap C(y) \neq \emptyset] \Rightarrow [y \succ_i v \succ_i x \Rightarrow \exists \mu > 0 : x+\mu t \sim_i v].$$

Producers

B1 Y is convex, closed and $0 \in \text{Int}\ Y$ (hence Y is star shaped)

B2 $Y \supset - R^{n+}$

Sums of sets

C1 $C_j(z_j) = \{z_j \mid \forall x_j \in X_j : z_j \succsim_j x_j\} \neq \emptyset$

$$\Rightarrow \{ \sum_{i \neq j} \text{Au}\ X_i + \text{Au}\ C_j(z_j)\} \cap Y = \emptyset$$

C2 $\Sigma\ X_i \cap Y \neq \emptyset$

C3 $\Sigma$ Cl Cone $X_i \cap$ Coneint $Y = \{0\}$.

C4 $\Sigma$ Cl Cone $X_i \cap - \Sigma$ Cl Cone $X_i = \{0\}$.

D Income distribution

For each p, such that $\max_{y \in Y} p\ y$ exists, there exist m continuous functions $\lambda_i(p)$ such that

$$\forall \mu > 0 : \lambda_i(p) = \lambda_i(\mu p)$$

$$\Sigma\ \lambda_i(p) = 1$$

$$\text{Int}\ X_i \cap \{x_i \mid \frac{p\ x_i}{\max_{y \in Y} p\ y} \leq \lambda_i(p)\} \neq \emptyset$$

$$\exists \tau > 0, \ \forall i, \forall p : \lambda_i(p) > \tau$$

This distribution functions might be based on a distribution of surplus income $\zeta_i(p)$ in a private ownership economy (see [ ] ), such that

$$\lambda_i(p) = \frac{p\, w_i + \zeta_i(p)}{\max p\, y}$$

for $\xi_i(p)$ continuous. The above conditions are full filled if we assume

$\{\text{Coneint } Y + (w_i)\} \cap \text{Int } X_i \neq \emptyset$, for each i.

Assumptions A1 - A4 are usual, apart from the assumption $0 \notin X_i$, necessary to garantee that $X_i^*$ is not empty. This condition (and C4 which requires the same for sums of $X_i$) is not as strong as it seems, because it could be constructed by a translation of the origin. A5 is a regularity condition, excluding that different preference sets converge to the same hyperplane, and therefore two dual preference sets have common boundary points. C1 encures that a satiable consumer is not satiated at a feasible allocation, C2 guarantees the existence of a feasible allocation, C3 rules out an "equilibrium" at infinity and by C4 the sum of all consumption sets, can be separated from (0).

4.3 Dual preference sets

With any set $C_i(x_i)$ is associated its dual $C_i^*(x_i)$, containing all prices at which no better point than $x_i$ is availabled an amount 1

$$C_i^*(x_i) = \{p_i \mid y \in C_i(x_i) \Rightarrow p_i\, x_i \geq 1\}.$$

Since $C_i(x_i)$ is closed and convex and does not contain the origin, $C_i^*(x_i)$ is <u>closed</u>, <u>convex</u>, and aureoled and $0 \notin C_i^*(x_i)$.

If $p_i$ is in the lower bound of $C_i^*(x_i)$, the plane $L(p)$ supports, or asymptotically supports $C_i(x_i)$.

Now let $P_i$ be the set of <u>all</u> prices such that $L(p)$ supports or asymptotically supports <u>one and only one</u> preference set, i.e. $P_i$ contains all prices that are in the lower bound of one and only one set $C_i^*(x_i)$

$$P_i = \{p_i \mid \exists\, x_i : p \in C^*(x_i) \text{ and } \lambda < 1 \Rightarrow \lambda\, p \notin C_i^*(x_i),$$

$$p \in C(y_i) \text{ and } y_i \sim x_i \Rightarrow \exists\, \mu < 1 : \mu\, p \in C(y_i)\}$$

Now assumption A5 guarantees that any p that is in some dual preference set, is in $P_i$, if it is not: $X_i^*$.

Hence no $L(p)$ can support or asymptotically support more dual sets, unless its supports $X_i$.

<u>Theorem</u> 4.3.1

$$\forall\, x_i \in X : C_i(x_i^*) \setminus X_i^* \subset P_i$$

<u>Proof</u>

a) If $p \in C_i^*(x_i) \setminus X_i^*$, $L(p)$ supports <u>some</u> preference set:

Since $p \in C_i^*(x_i)$, $L(p) \cap \operatorname{Int} C_i^*(x_i) = \emptyset$.

By the representation theorem $\succsim_i$ is representable by a continuous utility function $u_i(x_i)$.

Let $\overline{u}_i(p) = \inf\limits_{y_i \in L(p)} u_i(y) < u_i(x_i)$

Now $\overline{u}(p) = u(z_i)$ for some $x_i > z_i$ and $L(p)$ supports or asymptotically supports $C_i(z_i)$.

b) Suppose $L(p)$ supports (asymptotically) $C_i(x_i)$ and $C_i(y_i)$ for $x_i \succ y_i$ . Hence $p \in \text{Bnd}\ C_i^*(x_i)$ and $p \in \text{Bnd}\ C_i^*(y_i)$.
Since $C_i^*(x_i)$ is convex, there exists some hyperplane that supports $C_i^*(y_i)$ in $p$ : $\{q \mid q\ t = \alpha\}$, where $pt = \alpha$
Now suppose $\alpha \neq 0$ hence $q(\frac{1}{\alpha}\ t) = 1$ and hence $L(p)$ supports $C(x_i)$ in $x_i$ and then $L(p)$ cannot support $C(y_i)$, since $x_i \in C(y_i)$.

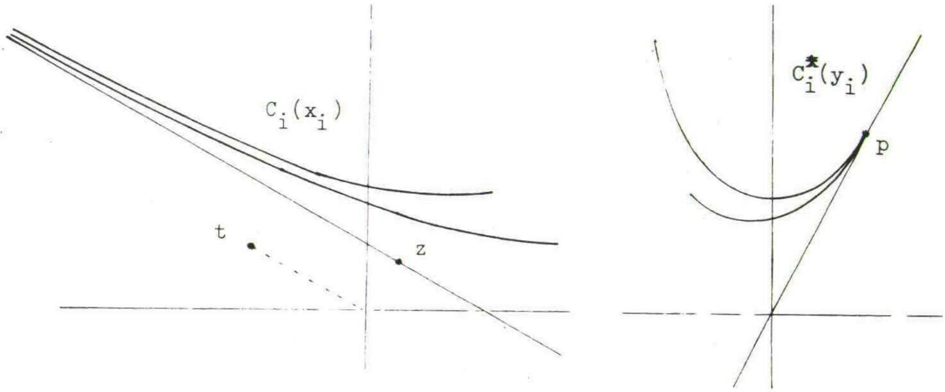



fig. 9

Hence $\alpha = 0$ and therefore $t \in \text{Cl Cone}\ C(x_i)$.
Let $z = \frac{p}{|p|^2}$ . Now $z \in L(p)$ and $\forall \mu > 1$: $\mu z + \lambda\ t \in C(x_i)$ for $\lambda$ sufficiently large.
But now also holds

$$\forall \varepsilon\ \exists \lambda : B_\varepsilon\ (z + \lambda t) \cap C(y) \neq \emptyset$$

but it is not true that for some $\lambda$

$$z + \lambda t \sim y$$

and that is in contradiction with assumption A5.
Note that it is not excluded that $L(p)$ supports both $X_i$ and some $C_i(x_i)$.

We can define $C^{*}$ as a correspondence

$$C^{*} : P \to R^{n}$$

where

$$C_i^{*}(p_i) = C_i^{*}(x_i) \text{ if } p \in C_i^{*}(x_i) \text{ and } \forall\ \lambda < 1 : p \notin C_i^{*}(x_i).$$

Theorem 4.3.2

The correspondence $C^{*} : P \to R^{n}$ is closed and l.s.c.

Proof

a) $C^{*}$ is closed in P

Let $p^{s} \to p^{o}$, $q^{s} \to q^{o}$ and $q^{s} \in C(p^{s})$, all points of P .
Suppose $q^{o} \notin C(p^{o})$. Hence $C(p^{o}) \subset C(q^{o})$. Choose $r \in \mathrm{Int}\ C(q^{o}) \setminus C(p^{o})$.
Now $C(p^{o}) \subset C(r) \subset C(q^{o})$ and $p^{o} \in \mathrm{Int}\ C(r)$ since $C(p^{o})$ cannot support more then one preference set, $q^{o} \notin C(r)$.
For some $s > n$, $p^{s} \in C(r)$ and for some $s > m$, $q^{s} \notin C(r)$. Hence if $s > n$ and $s > m$ $q^{s} \notin C(r)$, hence $q^{s} \notin C(r)$.

$C^{*}$ is l.s.c. in P

Let A be an open set, such that $C^{*}(p^{o}) \cap A \neq \emptyset$. Let $p^{o} \notin X_i^{*}$.
Now there exists $q^{o} \in C^{*}(p^{o}) \cap A$. $C(q^{o}) \subset C(p^{o})$ and $p^{o} \notin C(q^{o})$.
$P \setminus C(q^{o})$ is an open set and $r \in P \setminus C(q^{o}) \Rightarrow q^{o} \in C(r)$ hence $C(r) \cap A \neq \emptyset$. So $C^{*}$ is l.s.c.

## 4.4 Artificial satiation set

For consumers with satiation consumption $z_j$, we assumed

$$\{ \sum_{i \neq j} \mathrm{Au}\ X_i + \mathrm{Au}\ C_j(z_j)\} \cap Y = \emptyset.$$

For insatiable consumers we construct a set with the same properties,

so that this set cannot contain an equilibrium consumption.
By assumption

$$\Sigma \text{ Cone } X_i \cap - (\Sigma \text{ Cone } X_i) = \{0\}$$

and therefore

$$\{0\} \notin \text{Conv} \cup X_i$$

and hence

$$\cap X^{*} = (\cup X_i)^{*} \neq \emptyset.$$

Also by assumption

$$\Sigma \text{ Cl Cone } X_i \cap \text{Coneint } Y = \{0\}.$$

Therefore there exists $\lambda < 1$, such that $\lambda Y \cap \text{Conv} \cup X_i = \emptyset$, so there exists some hyperplane $L(r)$, which strictly separates both sets, hence

$$r \in (\text{Int} \cap X_i^{*}) \cap \text{Cone } Y^{*}$$

Hence there exists $\mu < 1$, such that $L(\mu r) \cap Y = \emptyset$, whereas $L(\mu r) \cap \text{Au } X_i$ is compact for each i.
Let j be an insatiable consumerand let $z_j$ be a most preferred point of $L(\mu r)$. Now $C_j(z_j) \cap Y = \emptyset$ and $C_j(z_j)$ is an artificial satiation set: obviously $L(\mu r)$ separates $\sum_{i \neq j} \text{Au } X_i + \text{Au } C_j(z_j)$ from Y, since $L(r)$ separates 0 and Au $X_i$.

With any satiated consumer can be associated a price vector $p(z_j)$, such that

$$p(z_j) \in \text{Int} \bigcap_{i \neq j} X_i^{*} \cap C_j^{*}(z_j).$$

For j insatiable, choose $p(z_j) = r$.

If j is satiable, by assumption

$$[\sum_{i \neq j} \mathrm{Au}\, X_i + \Sigma\, C_j(z_j)] \cap Y = \emptyset$$

hence

$$\mathrm{Int}\, [\sum_{i \neq j} \mathrm{Au}\, X_i + \Sigma\, C_j(z_j)]^{*} \cap Y^{*} \neq \emptyset.$$

Let q be a <u>interior</u> point of this set.

By <u>theorem</u> 2.5.1 $q = \varphi_i q_i \quad \Sigma \varphi_i \geq 1$ and $q_i \in X_i^{*}$, $q_j \in C_j^{*}(z_j)$.

For $\varphi = \min \varphi_i$, and $p(z_j) = \frac{1}{\varphi} q$, we have $p(z_j) \in \cap X_i^{*} \cap C_j^{*}(z_j)$.

## 4.5 <u>Existence</u>

We are now ready to prove the existence of an equilibrium for the economy defined in 4.1 and 4.2.
Obviously the result as such is not new, but the method of proof seems interesting and may be applicable for more general cases.

<u>Theorem</u>

An economy for which the assumptions at section 4.2 hold, has an equilibrium $\bar{x}_i$, $\bar{y}$, $\bar{p}$, such that

$$\bar{p}\,\bar{y} \geq \bar{p}\, z \qquad \text{for } z \in Y$$

$$\bar{p}\,\bar{x}_i = \lambda_i(\bar{p}) \geq \bar{p}\, v_i \qquad \text{for any } v_i \in C_i(\bar{x}_i)$$

$$\bar{y} = \Sigma\, \bar{x}_i.$$

<u>Proof</u>

Let $\cap C^{*}(z_j)$ be the intersection of the dual satiation sets, artificial or not. Since $\cap C_i^{*}(z_i) \supset \cap X_i^{*}$, this intersection has non empty interior.
Now choose a number m such that

1) $\forall i : \frac{1}{m} > \frac{1}{\tau_i}$ 3) $\cap X_i^* \cap \frac{1}{m} Y^* \neq \emptyset$

2) $\forall i : p(z_i) \in \frac{1}{m} Y^*$

Now let

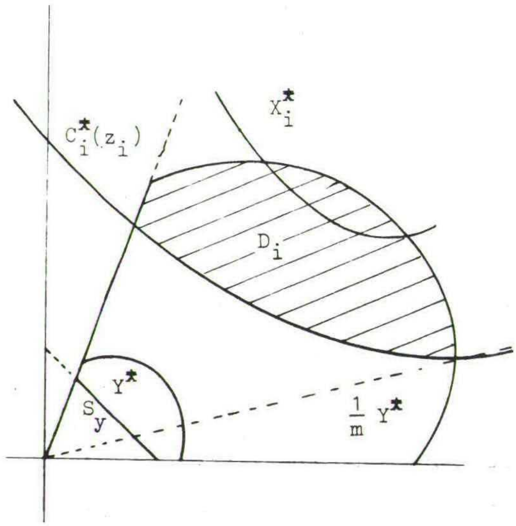
$$D_i = C^*(z_i) \cap \frac{1}{m} Y^*$$

$D_i$ is convex, compact. Let

$$\tilde{C}_i^*(p_i) = C_i^*(p_i) \cap D_i \qquad \text{for } p_i \in D_i \setminus X_i.$$

The correspondence $\tilde{C}_i^* : D_i \setminus X_i \to D_i$ is continuous, since $D_i$ is compact and $C_i^*$ is closed and lower semi continuous.

fig. 10




Let $S_+ = \{s \in R^n \mid s \geq 0 \text{ and } \Sigma\, s^k = 1\}$ be the set of nonpositive "unit" prices, which sum up to 1, and

$$S_y = S_+ \cap \text{Cone } Y^*$$

$S_y$ is non-empty, since $R^n \subset$ Coneint Y.

Further

$$S_i = S_y \cap \text{Cone } D_i$$

$$S = \cap S_i = S_y \cap (\cap \text{ Cone } D_i).$$

All sets S are non empty, convex and compact.

With any unit price $s \in S_y$, can be associated a real number $\alpha(s)$ and a production price $p(s)$ on the boundary of $Y^*$:

$$\alpha : S_y \to R, \text{ where } \alpha(s) = \max \{\alpha \mid \alpha s \in Y^*\}$$

$$p : S_y \to Y^*, \text{ where } p(s) = \alpha(s)s.$$

Both mappings are <u>continuous</u>, since $Y^*$ is a compact, convex set and $\alpha(s)$ is a <u>convex</u> function.

The income distribution function assigns an income $\lambda_i(p)$ to each individual i. Obviously $\lambda_i(p(s)) = \lambda_i(s) = \lambda_i(\alpha(s)s)$.

Now with any <u>production</u> price $p(s)$ can be associated an <u>individual</u> price $p_i$, by deflating the production price with the income. Hence we map the set of unit prices into the individual price-space:

$$p_i : S_y \to R^n, \text{ where } p_i(s) = \frac{1}{\lambda_i\ (p(s))}\ p(s) = \frac{\alpha(s)}{\lambda_i(s)}\ s.$$

Since both $\alpha(s)$ and $\lambda_i(s)$ are continuous, the function $p_i(s)$ is continuous. Note that $p_i(s) \geq 0$, since $\lambda_i(s) > 0$ and that $p_i(s) \notin X_i^*$ and $p_i(s) \in \frac{1}{m} Y^*$, since $\lambda_i(s) > m$.

We now define a correspondence $F_i : S_i \to S_i$, where

$$F_i(s) = \begin{cases} S_i \cap \text{Cone } \tilde{C}_i^*\ (p_i(s)) \text{ if } p_i(s) \in C_i^*(z_i) \\ \\ S_i \cap \text{Cone } \tilde{C}^*\ (z_i) \text{ if } p_i(s) \notin C_i^*(z_i) \end{cases}$$

This correspondence is <u>continuous</u> : The function $p_i(s)$ is continuous and maps $S_i$ into Cone $D_i$. Further $\tilde{C}^*_i(p_i)$ is continuous, hence $\tilde{C}^*_i(p_i(s))$ is continuous and therefore also the intersection of its cone with $S_i$. For points $p_i(s) \notin C_i(z)$, i.e. for individual prices which allow (artificial) satiation consumption, the correspondence gives $S_i \cap \text{Cone } C^*_i(z_i) = S_i$. We have

$$\forall\ s \in S_i : S_i \cap \text{Cone } (X^*_i \cap \frac{1}{m} Y^*) \subset F_i(s) \subset S_i$$

hence $F_i(s)$ is nowhere empty and $F_i(s)$ is a convex set.

Let $F : S \to S$ be the correspondence

$$F(s) = \cap F_i(s) \qquad \text{for } s \in S = \cap S_i$$

and this correspondence is continuous, not empty, since

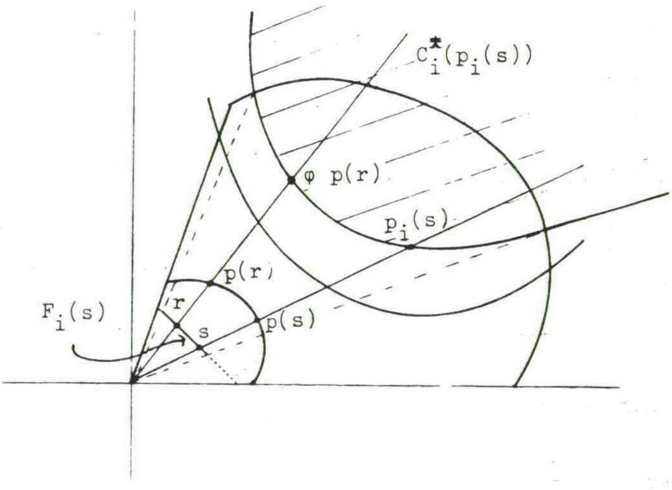



fig. 11

$\cap X_i^* \cap \frac{1}{m} Y^* \neq \emptyset$ and $F(s)$ is convex.

We further define a real valued fuction $g_i : S_i \times S_i \to R$, where

$$g_i(s,r) = \begin{cases} \min \{\varphi, \frac{1}{m} \mid \varphi\, p_i(r) \in C_i^*(p_i(s))\} \text{ if } p_i(s) \in C_i^*(z_i) \\ \\ \min \{\varphi, \frac{1}{m} \mid \varphi\, p(r) \in C_i^*\,(p_i(s))\} \text{ if } p_i(s) \notin C_i^*(z_i). \end{cases}$$

This function is continuous, convave in r over the set $F_i(s)$ and quasi concave in r over $S_i$.

Continuity : let $s^t, r^t \to s^o, r^o$ and $g_i(s^t, r^t) = \varphi^t \to \varphi^o$.

Since $s^t \to s^o$, we have $p_i(s^t) \to p_i(s^o)$. Let $q^t = \varphi^t p(r^t)$ hence $q^t \to q^o = \varphi^o p(r^o)$. Since by definition $q^t \in C_i^*(p_i(s^t))$, we have $q^o \in C_i^*(p_i(s^o))$, because of the closedeness of the correspondence $C_i^*$.

Now suppose $q^o \in \text{Int } C_i^*(p_i(s^o))$.

In this case there exist $\varphi < \varphi^o$ and $q < q^o$ such that $\varphi = g_i(s^o, r^o)$ and $q \in C_i^*(p_i(s^o))$. Let $B_\varepsilon(q)$ be an open neighbourhood of q.

Because of lower-semi-continuity of $C_i^*$, we would have, for $\varepsilon < \frac{1}{2}(\varphi^o - \varphi)$, $B_\varepsilon(q) \cap C_i^*(v) \neq \emptyset$ for any v in some neighbourhood of $p_i(s^o)$.

However, for t sufficiently large, $p_i(s^t)$ must be in such a neighbourhood and, since $q^t$ is in the boundary of $C_i^*(p_i(s^t))$, we have $B_\varepsilon(q) \cap C_i^*(p_i(s^t)) = \emptyset$, which is a contradiction.

The function is convex in r over $F_i(s)$ since both $Y^*$ and $C_i^*(p_i(s))$ are convex and the function measures the "distance" between these two sets.

Outside $F_i(s)$, the function is constant. We have

$$\frac{1}{m} > g_i(s,s) = \frac{1}{\lambda_i(s)} > 0 \qquad \text{if } p_i(s) \in C_i^*(z_i)$$

$$\frac{1}{m} > g_i(s,s) > \frac{1}{\lambda_i(s)} > 0 \qquad \text{if } p_i(s) \notin C_i^*(z_i).$$

Now let $f_i : S_i \times S_i \to R$, where

$$f_i\ (s,r) = \frac{1}{g_i\ (s,r)}\ .$$

Obviously this function is <u>continuous</u>, convex in r over $F_i(s)$ and quasi-convex in r over $S_i$.

Finally $f : S \times S \to R$, where

$$f(s,r) = \Sigma\ f_i(s,r).$$

This function is <u>continuous</u> and <u>convex</u> in r over F(s) and we have

$$f(s,s) = 1 = \Sigma\ \lambda_i(s) \text{ if } \forall i : p_i(s) \in C_i^*(z_i)$$

$$f(s,s) < 1 \qquad\qquad \text{if } \exists\ i : p_i(s) \notin C_i^*(z_i)$$

Let $\beta : S \to R$ and $H : S \to S$, where

$$\beta(s) = \max\ \{f(s,r)\ |\ r \in F(s)\}$$

$$H(s) = \{r\ |\ f(s,r) = \beta(s)\}.$$

By theorems 1 and 2 of Berge [ 1 ], page 121, 122, the function $\beta(s)$ is continuous on the compact convex set S.
By a slight generalisation[1] of Berges maximum theorem, the correspondence H is upper semi-continuous. Hence by Kakutani's fixed point theorem, there exists $\overline{s} \in S$, such that $\overline{s} \in H(\overline{s})$. In this fixed point, we have

$$f(\overline{s},\overline{s}) = \max_{r \in F(\overline{s})} f(\overline{s},r) = 1 \qquad \text{if } \forall\ i : p_i(\overline{s}) \in C_i^*(z_i)$$

$$< 1 \qquad \text{if } \exists\ i : p_i(\overline{s}) \notin C_i^*(z_i)$$

)[1] This theorem is given for one variable(f(s)). However for f(s,r) the proof is the same as the one given by Berge.

Case 1: $f(\bar{s},\bar{s}) = 1$

Now $C_i^*(p_i(\bar{s}))$ is the dual and $C_i(p_i(\bar{s}))$ is the original preference set and $p = \lambda_i(\bar{s}).\ p_i(\bar{s})$

$$p \in [\Sigma\ C_i(p_i(\bar{s}))]^* \cap\ Y^*.$$

There exists no interior point of $Y^*$, contained in $[\Sigma\ C_i(p_i(\bar{s}))]^*$, since for $p(r) \in Y^*$, we have $p(r) = f_i(\bar{s},r)\ p_i(r)$ and $\Sigma\ f_i(\bar{s},r) \leq 1$. Hence

$$p \notin \text{Int } Y^* \cap [\Sigma\ C_i(p_i(\bar{s}))]^*.$$

So $L(p(\bar{s}))$ supports both sets in some point $\bar{x}$, whereas $L(p_i(\bar{s}))$ supports $C_i(p_i(\bar{s}))$ in points $\bar{x}_i$, where $\Sigma\ \bar{x}_i = \bar{x}$. So $\bar{x}$ is an equilibrium and $p(\bar{s})$ is an equilibrium price.

Case 2: $f(\bar{s},\bar{s}) < 1$.

Now for some j, $p_j(\bar{s}) \notin C_j^*(z_j)$.

Since $p_j(z_j) \in \tilde{C}_j^*(z_j)$ and $p_j(z_j) \in \tilde{C}_j^*(p(\bar{s}))$, there exists $t \in F(\bar{s})$ such that $p(z_j) = \mu t$.

$$p(t) \in [\sum_{i \neq j}\ \text{Au}\ X_i + C_j(z_j)]^* \cap Y^*$$

and also

$$p(t) \in [\sum_{i \neq j}\ C_i(p_i(\bar{s})) + C_j(z_j)]^* \cap Y^*$$

and therefore

$$f(\bar{s},t) \geq 1$$

and this contradicts the hypothesis that

$$f(\bar{s},\bar{s}) = \max_{r \in F(\bar{s})}\ f(\bar{s},r) < 1.$$

This completes the proof of the theorem.

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